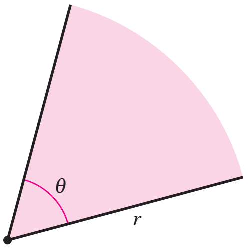
10.5 Area and Arc Length in Polar Coordinates

- **Find the area of a region bounded by a polar graph.**
- **Find the points of intersection of two polar graphs.**
- **Find the arc length of a polar graph.**
- **Find the area of a surface of revolution (polar form).**

Area of a Polar Region

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area. In Figure 10.48, note that the area of a circular sector of radius r is $\frac{1}{2}\theta r^2$, provided θ is measured in radians.



The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.
Figure 10.48

Consider the function $r = f(\theta)$, where f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 10.49(a). To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdot \cdot \cdot < \theta_{n-1} < \theta_n = \beta.$$

Then approximate the area of the region by the sum of the areas of the n sectors, as shown in Figure 10.49(b).

$$\text{Radius of } i\text{th sector} = f(\theta_i)$$

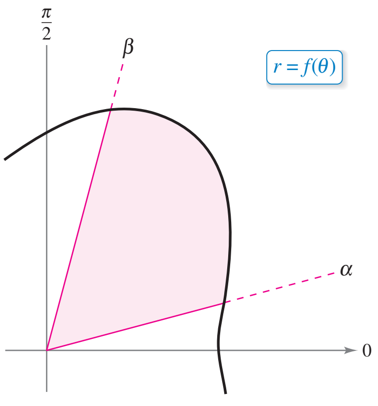
$$\text{Central angle of } i\text{th sector} = \frac{\beta - \alpha}{n} = \Delta\theta$$

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \Delta\theta [f(\theta_i)]^2$$

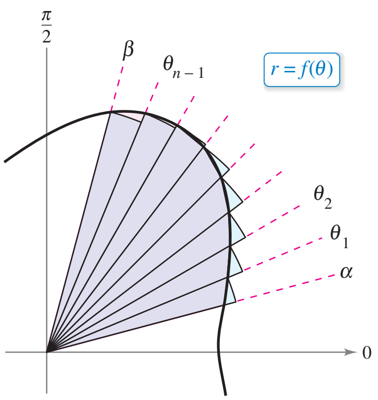
Taking the limit as $n \to \infty$ produces

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \, \Delta\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \end{aligned}$$

which leads to the next theorem.



(a)



(b)
Figure 10.49

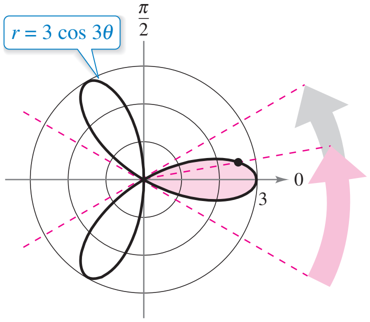
> **THEOREM 10.13 Area in Polar Coordinates**
>
> If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \le 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is
>
> $$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta. \qquad 0 < \beta - \alpha \le 2\pi \end{aligned}$$

You can use the formula in Theorem 10.13 to find the area of a region bounded by the graph of a continuous *nonpositive* function. The formula is not necessarily valid, however, when f takes on both positive *and* negative values in the interval $[\alpha, \beta]$.

EXAMPLE 1 Finding the Area of a Polar Region

See LarsonCalculus.com for an interactive version of this type of example.

Find the area of one petal of the rose curve $r = 3 \cos 3\theta$.



The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.
Figure 10.50

Solution In Figure 10.50, you can see that the petal on the right is traced as θ increases from $-\pi/6$ to $\pi/6$. So, the area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2\,d\theta$$ Use formula for area in polar coordinates.

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2}\,d\theta$$ Power-reducing formula

$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6}$$

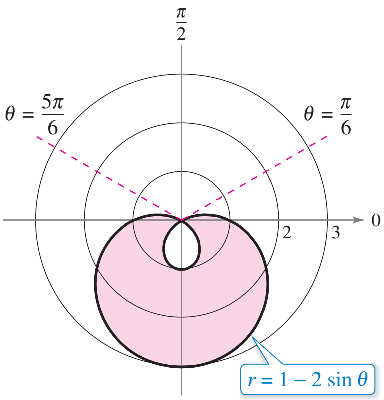
$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{3\pi}{4}.$$

To find the area of the region lying inside all three petals of the rose curve in Example 1, you could *not* simply integrate between 0 and 2π. By doing this, you would obtain $9\pi/2$, which is twice the area of the three petals. The duplication occurs because the rose curve is traced twice as θ increases from 0 to 2π.

EXAMPLE 2 Finding the Area Bounded by a Single Curve

Find the area of the region lying between the inner and outer loops of the limaçon $r = 1 - 2 \sin \theta$.



The area between the inner and outer loops is approximately 8.34.
Figure 10.51

Solution In Figure 10.51, note that the inner loop is traced as θ increases from $\pi/6$ to $5\pi/6$. So, the area inside the *inner loop* is

$$A_1 = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\,d\theta$$ Use formula for area in polar coordinates.

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 4\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 4\sin\theta + 4\left(\frac{1 - \cos 2\theta}{2}\right)\right]d\theta$$ Power-reducing formula

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\sin\theta - 2\cos\theta)\,d\theta$$ Simplify.

$$= \frac{1}{2}\Big[3\theta + 4\cos\theta - \sin 2\theta\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$$

$$= \pi - \frac{3\sqrt{3}}{2}.$$

In a similar way, you can integrate from $5\pi/6$ to $13\pi/6$ to find that the area of the region lying inside the *outer loop* is $A_2 = 2\pi + \left(3\sqrt{3}/2\right)$. The area of the region lying between the two loops is the difference of A_2 and A_1.

$$A = A_2 - A_1 = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3} \approx 8.34$$

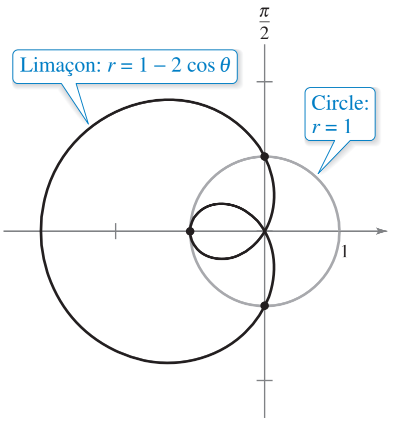
Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs. For example, consider the points of intersection of the graphs of

$$r = 1 - 2\cos\theta \quad \text{and} \quad r = 1$$

as shown in Figure 10.52. As with rectangular equations, you can attempt to find the points of intersection by solving the two equations simultaneously, as shown.

$$r = 1 - 2\cos\theta \qquad \text{First equation}$$
$$1 = 1 - 2\cos\theta \qquad \text{Substitute } r = 1 \text{ from 2nd equation into 1st equation.}$$
$$\cos\theta = 0 \qquad \text{Simplify.}$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \text{Solve for } \theta.$$



Three points of intersection: $(1, \pi/2)$, $(-1, 0)$, and $(1, 3\pi/2)$
Figure 10.52

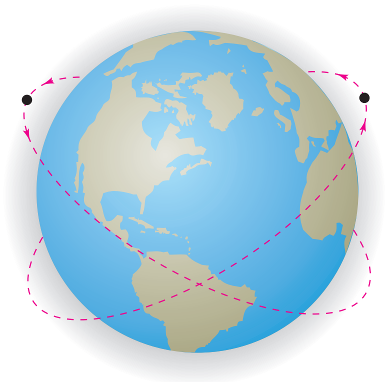
The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$. From Figure 10.52, however, you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously. (This is one reason why you should sketch a graph when finding the area of a polar region.) The reason the third point was not found is that it does not occur with the same coordinates in the two graphs. On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of

$$r = 1 - 2\cos\theta$$

the point occurs with coordinates $(-1, 0)$.

In addition to solving equations simultaneously and sketching a graph, note that because the pole can be represented by $(0, \theta)$, where θ is *any* angle, you should check separately for the pole when finding points of intersection.

You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 10.53. The satellites will not collide as long as they reach the points of intersection at different times (θ-values). Collisions will occur only at the points of intersection that are "simultaneous points"—those that are reached at the same time (θ-value).



The paths of satellites can cross without causing a collision.
Figure 10.53

FOR FURTHER INFORMATION For more information on using technology to find points of intersection, see the article "Finding Points of Intersection of Polar-Coordinate Graphs" by Warren W. Esty in *Mathematics Teacher*. To view this article, go to *MathArticles.com.*

EXAMPLE 3 Finding the Area of a Region Between Two Curves

Find the area of the region common to the two regions bounded by the curves.

$$r = -6\cos\theta \qquad \text{Circle}$$

and

$$r = 2 - 2\cos\theta. \qquad \text{Cardioid}$$

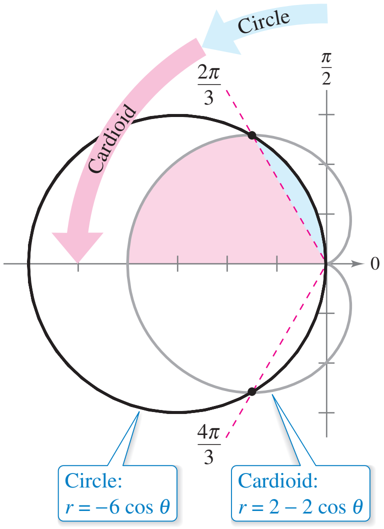


Figure 10.54

Solution Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 10.54. The blue shaded region lies between the circle and the radial line

$$\theta = \frac{2\pi}{3}.$$

Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region. The region that is shaded red is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid. So, you can find the area of this second region by integrating between $2\pi/3$ and π. The sum of these two integrals gives the area of the common region lying *above* the radial line $\theta = \pi$.

$$\frac{A}{2} = \underbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2\, d\theta}_{\text{Region between circle and radial line } \theta = 2\pi/3} + \underbrace{\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2\, d\theta}_{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

$$= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta\, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= 9\int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta)\, d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi}$$

$$= 9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{5\pi}{2}$$

Finally, multiplying by 2, you can conclude that the total area is

$$5\pi \approx 15.7. \qquad \text{Area of region inside circle and cardioid}$$

To check the reasonableness of this result, note that the area of the circular region is

$$\pi r^2 = 9\pi. \qquad \text{Area of circle}$$

So, it seems reasonable that the area of the region lying inside the circle and the cardioid is 5π.

To see the benefit of polar coordinates for finding the area in Example 3, consider the integral below, which gives the comparable area in rectangular coordinates.

$$\frac{A}{2} = \int_{-4}^{-3/2} \sqrt{2\sqrt{1 - 2x} - x^2 - 2x + 2}\, dx + \int_{-3/2}^{0} \sqrt{-x^2 - 6x}\, dx$$

Use the integration capabilities of a graphing utility to show that you obtain the same area as that found in Example 3.

Arc Length in Polar Form

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 85.)

REMARK When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration. For instance, the rose curve $r = \cos 3\theta$ is traced out once on the interval $0 \le \theta \le \pi$, but is traced out twice on the interval $0 \le \theta \le 2\pi$.

THEOREM 10.14 Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 4 Finding the Length of a Polar Curve

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$$r = f(\theta) = 2 - 2\cos\theta$$

as shown in Figure 10.55.

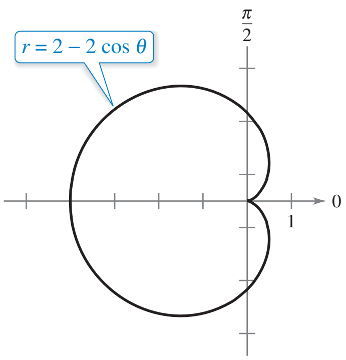


Figure 10.55

Solution Because $f'(\theta) = 2\sin\theta$, you can find the arc length as follows.

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \qquad \text{Formula for arc length of a polar curve}$$

$$= \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta \qquad \text{Simplify.}$$

$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta \qquad \text{Trigonometric identity}$$

$$= 4\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \text{ for } 0 \le \theta \le 2\pi$$

$$= 8\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 8(1 + 1)$$

$$= 16$$

Using Figure 10.55, you can determine the reasonableness of this answer by comparing it with the circumference of a circle. For example, a circle of radius $\frac{5}{2}$ has a circumference of

$$5\pi \approx 15.7.$$

Note that in the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2\frac{\theta}{2}} = \sqrt{2}\sin\frac{\theta}{2}$$

rather than

$$\sqrt{2\sin^2\frac{\theta}{2}} = \sqrt{2}\left|\sin\frac{\theta}{2}\right|$$

because $\sin(\theta/2) \ge 0$ for $0 \le \theta \le 2\pi$.

Area of a Surface of Revolution

The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions given in Theorem 10.9, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

REMARK When using Theorem 10.15, check to see that the graph of $r = f(\theta)$ is traced only once on the interval $\alpha \le \theta \le \beta$. For example, the circle $r = \cos\theta$ is traced only once on the interval $0 \le \theta \le \pi$.

THEOREM 10.15 Area of a Surface of Revolution

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ — About the polar axis

2. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ — About the line $\theta = \frac{\pi}{2}$

EXAMPLE 5 Finding the Area of a Surface of Revolution

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 10.56.

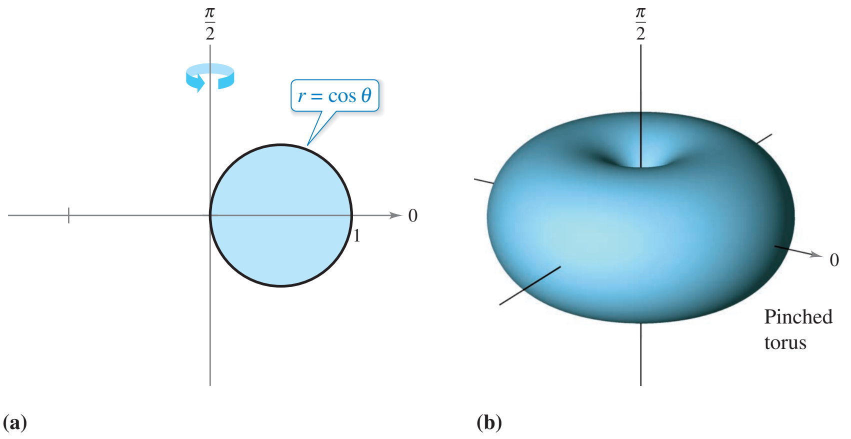


Figure 10.56

Solution Use the second formula in Theorem 10.15 with $f'(\theta) = -\sin\theta$. Because the circle is traced once as θ increases from 0 to π, you have

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta \quad \text{Formula for area of a surface of revolution}$$

$$= 2\pi \int_{0}^{\pi} \cos\theta(\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi \int_{0}^{\pi} \cos^2\theta\,d\theta \quad \text{Trigonometric identity}$$

$$= \pi \int_{0}^{\pi} (1 + \cos 2\theta)\,d\theta \quad \text{Trigonometric identity}$$

$$= \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

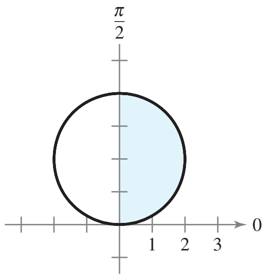
$$= \pi^2.$$

10.5 Exercises

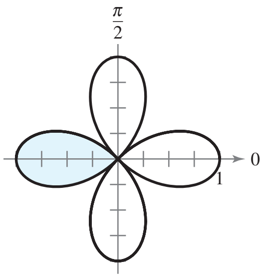
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Area of a Polar Region **In Exercises 1–4, write an integral that represents the area of the shaded region of the figure. Do not evaluate the integral.**

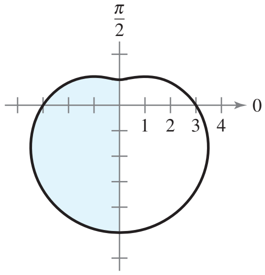
1. $r = 4 \sin \theta$



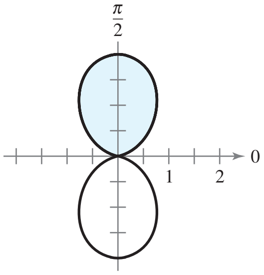
2. $r = \cos 2\theta$



3. $r = 3 - 2 \sin \theta$



4. $r = 1 - \cos 2\theta$



Finding the Area of a Polar Region **In Exercises 5–16, find the area of the region.**

5. Interior of $r = 6 \sin \theta$

6. Interior of $r = 3 \cos \theta$

7. One petal of $r = 2 \cos 3\theta$

8. One petal of $r = 4 \sin 3\theta$

9. One petal of $r = \sin 2\theta$

10. One petal of $r = \cos 5\theta$

11. Interior of $r = 1 - \sin \theta$

12. Interior of $r = 1 - \sin \theta$ (above the polar axis)

13. Interior of $r = 5 + 2 \sin \theta$

14. Interior of $r = 4 - 4 \cos \theta$

15. Interior of $r^2 = 4 \cos 2\theta$

16. Interior of $r^2 = 6 \sin 2\theta$

Finding the Area of a Polar Region **In Exercises 17–24, use a graphing utility to graph the polar equation. Find the area of the given region analytically.**

17. Inner loop of $r = 1 + 2 \cos \theta$

18. Inner loop of $r = 2 - 4 \cos \theta$

19. Inner loop of $r = 1 + 2 \sin \theta$

20. Inner loop of $r = 4 - 6 \sin \theta$

21. Between the loops of $r = 1 + 2 \cos \theta$

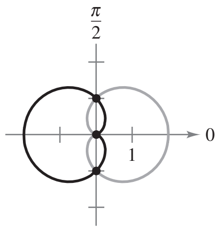
22. Between the loops of $r = 2(1 + 2 \sin \theta)$

23. Between the loops of $r = 3 - 6 \sin \theta$

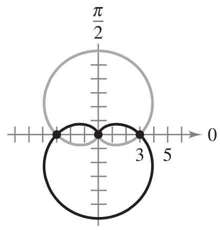
24. Between the loops of $r = \frac{1}{2} + \cos \theta$

Finding Points of Intersection **In Exercises 25–32, find the points of intersection of the graphs of the equations.**

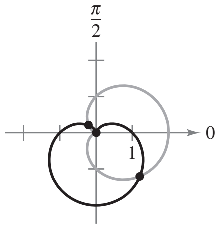
25. $r = 1 + \cos \theta$
$r = 1 - \cos \theta$



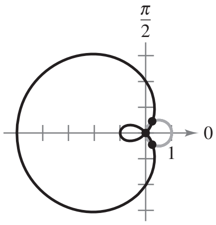
26. $r = 3(1 + \sin \theta)$
$r = 3(1 - \sin \theta)$



27. $r = 1 + \cos \theta$
$r = 1 - \sin \theta$



28. $r = 2 - 3 \cos \theta$
$r = \cos \theta$



29. $r = 4 - 5 \sin \theta$
$r = 3 \sin \theta$

30. $r = 3 + \sin \theta$
$r = 2 \csc \theta$

31. $r = \dfrac{\theta}{2}$
$r = 2$

32. $\theta = \dfrac{\pi}{4}$
$r = 2$

Writing **In Exercises 33 and 34, use a graphing utility to graph the polar equations and approximate the points of intersection of the graphs. Watch the graphs as they are traced in the viewing window. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.**

33. $r = \cos \theta$
$r = 2 - 3 \sin \theta$

34. $r = 4 \sin \theta$
$r = 2(1 + \sin \theta)$

Finding the Area of a Polar Region Between Two Curves **In Exercises 35–42, use a graphing utility to graph the polar equations. Find the area of the given region analytically.**

35. Common interior of $r = 4 \sin 2\theta$ and $r = 2$

36. Common interior of $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$

37. Common interior of $r = 3 - 2 \sin \theta$ and $r = -3 + 2 \sin \theta$

38. Common interior of $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$

39. Common interior of $r = 4 \sin \theta$ and $r = 2$

40. Common interior of $r = 2 \cos \theta$ and $r = 2 \sin \theta$

41. Inside $r = 2 \cos \theta$ and outside $r = 1$

42. Inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$

Finding the Area of a Polar Region Between Two Curves **In Exercises 43–46, find the area of the region.**

43. Inside $r = a(1 + \cos\theta)$ and outside $r = a\cos\theta$

44. Inside $r = 2a\cos\theta$ and outside $r = a$

45. Common interior of $r = a(1 + \cos\theta)$ and $r = a\sin\theta$

46. Common interior of $r = a\cos\theta$ and $r = a\sin\theta$, where $a > 0$

47. Antenna Radiation

The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by $r = a\cos^2\theta$.

(a) Convert the polar equation to rectangular form.

(b) Use a graphing utility to graph the model for $a = 4$ and $a = 6$.

(c) Find the area of the geographical region between the two curves in part (b).

48. Area The area inside one or more of the three interlocking circles

$$r = 2a\cos\theta, \quad r = 2a\sin\theta, \quad \text{and} \quad r = a$$

is divided into seven regions. Find the area of each region.

49. Conjecture Find the area of the region enclosed by

$$r = a\cos(n\theta)$$

for $n = 1, 2, 3, \ldots$. Use the results to make a conjecture about the area enclosed by the function when n is even and when n is odd.

50. Area Sketch the strophoid

$$r = \sec\theta - 2\cos\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Convert this equation to rectangular coordinates. Find the area enclosed by the loop.

Finding the Arc Length of a Polar Curve **In Exercises 51–56, find the length of the curve over the given interval.**

	Polar Equation	Interval
51.	$r = 8$	$0 \le \theta \le 2\pi$
52.	$r = a$	$0 \le \theta \le 2\pi$
53.	$r = 4\sin\theta$	$0 \le \theta \le \pi$
54.	$r = 2a\cos\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
55.	$r = 1 + \sin\theta$	$0 \le \theta \le 2\pi$
56.	$r = 8(1 + \cos\theta)$	$0 \le \theta \le 2\pi$

Finding the Arc Length of a Polar Curve **In Exercises 57–62, use a graphing utility to graph the polar equation over the given interval. Use the integration capabilities of the graphing utility to approximate the length of the curve accurate to two decimal places.**

	Polar Equation	Interval
57.	$r = 2\theta$	$0 \le \theta \le \frac{\pi}{2}$
58.	$r = \sec\theta$	$0 \le \theta \le \frac{\pi}{3}$
59.	$r = \frac{1}{\theta}$	$\pi \le \theta \le 2\pi$
60.	$r = e^{\theta}$	$0 \le \theta \le \pi$
61.	$r = \sin(3\cos\theta)$	$0 \le \theta \le \pi$
62.	$r = 2\sin(2\cos\theta)$	$0 \le \theta \le \pi$

Finding the Area of a Surface of Revolution **In Exercises 63–66, find the area of the surface formed by revolving the curve about the given line.**

	Polar Equation	Interval	Axis of Revolution
63.	$r = 6\cos\theta$	$0 \le \theta \le \frac{\pi}{2}$	Polar axis
64.	$r = a\cos\theta$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
65.	$r = e^{a\theta}$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
66.	$r = a(1 + \cos\theta)$	$0 \le \theta \le \pi$	Polar axis

Finding the Area of a Surface of Revolution **In Exercises 67 and 68, use the integration capabilities of a graphing utility to approximate, to two decimal places, the area of the surface formed by revolving the curve about the polar axis.**

67. $r = 4\cos 2\theta, \quad 0 \le \theta \le \frac{\pi}{4}$

68. $r = \theta, \quad 0 \le \theta \le \pi$

WRITING ABOUT CONCEPTS

69. Points of Intersection Explain why finding points of intersection of polar graphs may require further analysis beyond solving two equations simultaneously.

70. Area of a Surface of Revolution Give the integral formulas for the area of the surface of revolution formed when the graph of $r = f(\theta)$ is revolved about

(a) the polar axis.

(b) the line $\theta = \pi/2$.

71. Area of a Region For each polar equation, sketch its graph, determine the interval that traces the graph only once, and find the area of the region bounded by the graph using a geometric formula and integration.

(a) $r = 10\cos\theta$ (b) $r = 5\sin\theta$

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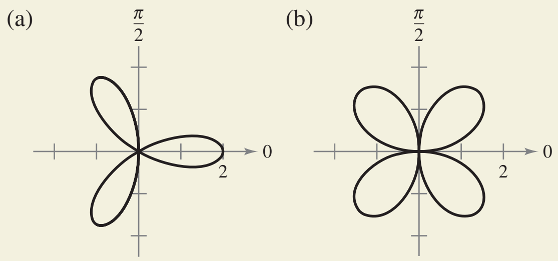
72. **HOW DO YOU SEE IT?** Which graph, traced out only once, has a larger arc length? Explain your reasoning.



73. Surface Area of a Torus Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5 \sec \theta$.

74. Surface Area of a Torus Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b \sec \theta$, where $0 < a < b$.

75. Approximating Area Consider the circle $r = 8 \cos \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(e) Do the results of part (d) depend on the radius of the circle? Explain.

76. Approximating Area Consider the circle $r = 3 \sin \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

77. Conic What conic section does the polar equation $r = a \sin \theta + b \cos \theta$ represent?

78. Area Find the area of the circle given by

$$r = \sin \theta + \cos \theta.$$

Check your result by converting the polar equation to rectangular form, then using the formula for the area of a circle.

79. Spiral of Archimedes The curve represented by the equation $r = a\theta$, where a is a constant, is called the spiral of Archimedes.

(a) Use a graphing utility to graph $r = \theta$, where $\theta \geq 0$. What happens to the graph of $r = a\theta$ as a increases? What happens if $\theta \leq 0$?

(b) Determine the points on the spiral $r = a\theta$ $(a > 0, \theta \geq 0)$, where the curve crosses the polar axis.

(c) Find the length of $r = \theta$ over the interval $0 \leq \theta \leq 2\pi$.

(d) Find the area under the curve $r = \theta$ for $0 \leq \theta \leq 2\pi$.

80. Logarithmic Spiral The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a **logarithmic spiral.** The figure shows the graph of $r = e^{\theta/6}$, $-2\pi \leq \theta \leq 2\pi$. Find the area of the shaded region.

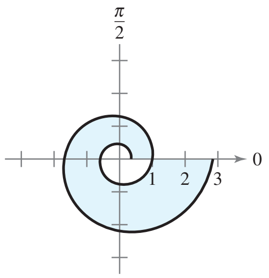


Figure for 80

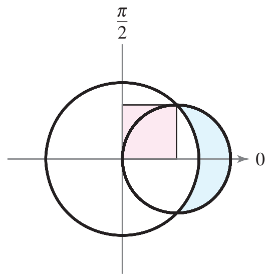


Figure for 81

81. Area The larger circle in the figure is the graph of $r = 1$. Find the polar equation of the smaller circle such that the shaded regions are equal.

82. Folium of Descartes A curve called the **folium of Descartes** can be represented by the parametric equations

$$x = \frac{3t}{1 + t^3} \quad \text{and} \quad y = \frac{3t^2}{1 + t^3}.$$

(a) Convert the parametric equations to polar form.

(b) Sketch the graph of the polar equation from part (a).

(c) Use a graphing utility to approximate the area enclosed by the loop of the curve.

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If $f(\theta) > 0$ for all θ and $g(\theta) < 0$ for all θ, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ do not intersect.

84. If $f(\theta) = g(\theta)$ for $\theta = 0$, $\pi/2$, and $3\pi/2$, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ have at least four points of intersection.

85. Arc Length in Polar Form Use the formula for the arc length of a curve in parametric form to derive the formula for the arc length of a polar curve.